

Electron transmission and phase time in semiconductor superlattices

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We discuss the time spent by an electron propagating through a finite periodic system such as a semiconductor superlattice. The relation between dwell-time and phase-time is outlined. The envelopes of phase-time at maximum and minimum transmission are derived, and it is shown that the peaks and valleys of phase-time can be well described by parameters fitted at the extrema. For a many-period system this covers most of the allowed band. Comparison is made to direct numerical solutions of the time-dependent Schrödinger equation by Veenstra et al. {cond-mat/0411118} who compared systems with and without addition of an anti-reflection coating (ARC). With an ARC, the time delay is consistent with propagation at the Bloch velocity of the periodic system, which significantly reduces the time delay, in addition to increasing the transmissivity.

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I. INTRODUCTION

The question of “how long does it take for a non-relativistic particle to cross a barrier”, or in general any potential, is a contentious one in quantum physics [1, 2, 3]. There is for example the Hartmann effect [4], according to which a transmitted particle might be found before it has reached the potential. Since 1990 the subject has taken on renewed interest in the context of electrons in semiconductor superlattices (SL) [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. It is commonly assumed that such propagation is ballistic, with a mean free path larger than the device dimensions. But because electrons can scatter from lattice vibrations, it is of interest to know how long the electron is exposed to such interactions. Many theories have been put forward as to how the time of passage should be defined and measured; among them dwell-time and phase-time are the best recognized. Veenstra et al. [18] studied time dependence of propagation by direct numerical solution of the time-dependent Schrödinger equation (TDSE), using gaussian incident wave packets. Their results agreed well with phase-time. In this contribution, based on a talk given at Theory-Canada 3 in June 2007, we explain why phase-time should be applicable in the context of semiconductor superlattices.

II. TRANSFER MATRIX METHOD

Assuming ballistic transport, a conduction band electron in a potential cell of arbitrary shape on $a < x < b$ satisfies the Schrödinger equation with a material-

dependent effective mass:

$$\begin{aligned} m^*(x) \frac{d}{dx} \left[\frac{d\psi(x)}{m^*(x)dx} \right] + \frac{2m^*(x)}{\hbar^2} [E - V(x)]\psi(x) &= 0 \\ k^2(x) &= \frac{2m^*(x)}{\hbar^2} [E - V(x)] \\ \Psi(x, t) &= e^{-iEt/\hbar} \psi(x). \end{aligned} \quad (1)$$

The approximations leading to eq.1 are fully explained in Bastard's monograph [17]. At a layer boundary, $\psi(x)$ and $\psi' \equiv (\hbar/m^*)d\psi/dx$ are continuous, to conserve flux. It is convenient to match $\psi(x)$ to plane waves, normalized to unit flux, at the boundaries of the unit cell $a < x < b$ [19]

$$\begin{aligned} \psi_L(x) &= \frac{c_L}{\sqrt{v_L}} e^{ik_L(x-a)} + \frac{d_L}{\sqrt{v_L}} e^{-ik_L(x-a)} , \quad x < a, \\ \psi_R(x) &= \frac{c_R}{\sqrt{v_R}} e^{ik_R(x-b)} + \frac{d_R}{\sqrt{v_R}} e^{-ik_R(x-b)} , \quad x > b \end{aligned} \quad (2)$$

At either end, ($B = L, R$) k_B is the wave number outside the periodic system, and $v_B = \hbar k_B/m_B^*$ is the velocity. We will consider systems without bias, so $k_L = k_R = k$, etc., but it will be convenient to retain the indices in order to know where various terms come from. The transfer matrix relates the coefficients on either side:

$$\begin{pmatrix} c_L \\ d_L \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} c_R \\ d_R \end{pmatrix} \quad \text{where} \\ M = \begin{pmatrix} 1/t & r^*/t^* \\ r/t & 1/t^* \end{pmatrix} \quad (3)$$

in terms of the reflection and transmission amplitudes $r(k), t(k)$. It contains all the information about scattering from that cell. One can show [20] that

$$M\sigma_z M^\dagger = \sigma_z \Rightarrow |c|^2 - |d|^2 = \text{constant}. \quad (4)$$

which says that the probability flux is preserved by the action of M , and which implies $\det M = 1$.

It is convenient to use Kard's parameterization [21] of M . In an allowed band we write

$$\begin{aligned} M_{11} &= \cos \phi - i \sin \phi \cosh \mu = 1/t = M_{22}^* \\ M_{21} &= -i e^{i\chi} \sin \phi \sinh \mu = r/t = M_{12}^*. \end{aligned} \quad (5)$$

At a given energy, the Bloch phase is determined by $\text{Tr}M = 2 \cos \phi$; the impedance parameter from the ratio of $|M_{21}|/\text{Im}M_{11} = \tanh \mu$, and the asymmetry parameter χ from the phase of $M_{21}/M_{12} = \exp(2i\chi)$. ($\chi = 0$ for a reflection-symmetric potential cells. For the most part this paper is restricted to the symmetric case.) For a simple square barrier-cell, e^μ is the ratio of average velocity outside to inside. Across an allowed band, ϕ increases by π , while μ is quite constant across most of the band, diverging at the band edges. An example of this behaviour is shown in Fig. 1. taken from [18]. Panel (a) show $\cos \phi$ for a square barrier cell, and for a phase-shift equivalent gaussian barrier cell. Panel (b) shows the corresponding values for μ . The dotted lines in each panel are the parameters of the single-layer ARC cell. The slope of $\cos \phi_A$ is about half that of $\cos \phi$, while their Bragg points coincide. The rule of thumb for ARC's in optics would make $\mu_A = \mu/2$.

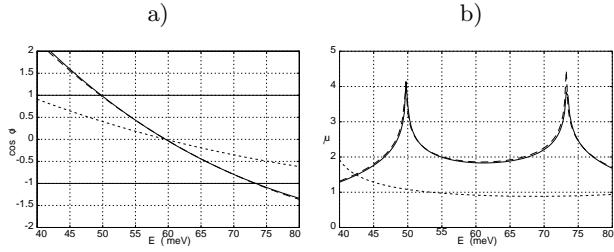


FIG. 1: Comparison of (a) Bloch phases of a square barrier cell (dash line) and a gaussian cell (solid line); also shown is $\cos \phi_A$ of a single-cell ARC (dotted line). (b) same for impedance parameters μ , μ_A .

In the Kard representation, transfer matrix factorizes:

$$\begin{aligned} M &= \cos \phi - i \sin \phi [\cosh \mu + \sinh \mu (\cos \chi \sigma_x + \sin \chi \sigma_y)] \\ &= e^{-i(\chi/2)\sigma_z} e^{(\mu/2)\sigma_x} e^{-i\phi\sigma_z} e^{-(\mu/2)\sigma_x} e^{i(\chi/2)\sigma_z} \end{aligned} \quad (6)$$

The matrix of eigenstates is easily seen to be

$$U = e^{-i(\chi/2)\sigma_z} e^{(\mu/2)\sigma_x}$$

Substitution into eq. 6 gives

$$\begin{aligned} MU &= U e^{-i\phi\sigma_z}, \\ U &= \begin{pmatrix} e^{-i\chi/2} \cosh \mu/2 & e^{-i\chi/2} \sinh \mu/2 \\ e^{i\chi/2} \sinh \mu/2 & e^{i\chi/2} \cosh \mu/2 \end{pmatrix} \end{aligned} \quad (7)$$

The eigenvalues are $e^{-i\phi}$ and $e^{+i\phi}$. Physically, the eigenstates are the Bloch waves of an infinite periodic array

composed of cells, each described by M . Mathematically, M is a hyperbolic rotation of a two-dimensional Dirac spinor, about an invariant axis whose polar angles are (μ, χ) [20].

The usefulness of the transfer matrix for a periodic system arises from the property

$$\begin{aligned} M^{(N)} &= M^N(\phi, \mu, \chi) = M(N\phi, \mu, \chi) \Rightarrow \\ \frac{1}{t_N} &= \cos N\phi - i \sin N\phi \cosh \mu. \end{aligned} \quad (8)$$

Hence the transmission probability for N -cells

$$|t_N|^2 = [1 + \sin^2 N\phi \sinh^2 \mu]^{-1} \geq 1/\cosh^2 \mu \quad (9)$$

shows narrow peaks determined by $N\phi(E) = m\pi$, $m = 1, 2, \dots, N-1$. Between these peaks the $|t_N|^2$ touches the envelope of minima, $1/\cosh^2 \mu$. At the center of an allowed band, the envelope of minima crosses the curve $|t_1|^2$.

In forbidden bands, the Bloch phase $\phi \rightarrow p\pi + i\theta$ acquires an imaginary part; as a result the transmission goes rapidly to zero. Pacher et al. [22] used this property to design an electron band-pass filter. Further, by adding a quarter-wave cell (anti-reflection coating, or ARC) at each end of the periodic array [23, 24], they were able to increase the average transmission within the band from about 25% to about 75%. In a series of papers [20, 25, 26, 27], some of us showed how to design an ARC which gives optimal transmission within a given miniband, by adding suitably configured potential cells on each end of a periodic array. Without an ARC, electrons which are transmitted do so via narrow resonances as in eq. 9, so one expects a significant time delay. With an ARC, the incident plane wave is transformed into a Bloch wave of the periodic potential, and travels at the Bloch velocity. This should reduce the transit time.

III. TIME DELAY

A. Dwell time and phase time

As already mentioned, the subject of time-delay in scattering is controversial. Razavy's book [3] is a useful introduction. Nussenzweig [5, 6] has argued that dwell-time is the best founded, for description of wave packet scattering. First we explain two of these theories.

Classically,

$$dt = \frac{dx}{v(x)} = \frac{dx}{\hbar k(x)/m}. \quad (10)$$

For a plane wave normalized to unit flux, as in eq. 2,

$$|\psi_0(x, E)|^2 = \frac{m^*}{\hbar k} = \frac{1}{v_{cl}}. \quad (11)$$

Dwell-time delay is defined to be the difference

$$\tau_D = \int_{x_L}^{x_R} [|\psi(x, E)|^2 - |\psi_0(x, E)|^2] dx. \quad (12)$$

Asymptotically, the scattering wave function, for waves incident from the left, is usually written

$$\begin{aligned}\psi(x, E) &= \frac{1}{\sqrt{v_L}} [e^{ik_L x} + r e^{-ik_L x}] \quad , \quad x < x_L \\ &= \frac{1}{\sqrt{v_R}} t e^{ik_R x} \quad , \quad x > x_R , \\ \text{where} \quad t &= |t| e^{i\eta} \quad , \quad r = |r| e^{i\delta} \end{aligned}\quad (13)$$

define the scattering phase shifts.

Following Smith [29], the Schrödinger equation gives the identity

$$\begin{aligned}\psi &= (H - E) \frac{\partial \psi}{\partial E} \quad \Rightarrow \\ \psi^* \psi &= -\frac{\hbar^2}{2m} \operatorname{Re} \left\{ \frac{\partial}{\partial x} \left[\psi^* \frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial E} \right) - \frac{\partial \psi}{\partial E} \frac{\partial \psi^*}{\partial x} \right] \right\} , \\ \int_{x_L}^{x_R} \psi^* \psi dx &= -\frac{\hbar^2}{2m} \operatorname{Re} \left[\psi^* \frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial E} \right) - \frac{\partial \psi}{\partial E} \frac{\partial \psi^*}{\partial x} \right] \Big|_{x_L}^{x_R} \end{aligned}\quad (14)$$

For a potential on $a < x < b$, $x_L < a$ and $x_R > b$ are positions at which a position measurement could be carried out. Taking only the scattering wave function in eq. 12,

$$\begin{aligned}-\frac{\hbar^2}{2m} \operatorname{Re} \left[\psi^* \frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial E} \right) - \frac{\partial \psi}{\partial E} \frac{\partial \psi^*}{\partial x} \right]_{x_R} &= \left(\frac{x_R}{v_R} + \hbar \frac{\partial \eta}{\partial E} \right) |t| \\ -\frac{\hbar^2}{2m} \operatorname{Re} \left[\psi^* \frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial E} \right) - \frac{\partial \psi}{\partial E} \frac{\partial \psi^*}{\partial x} \right]_{x_L} &= \\ \frac{x_L}{v_L} (1 + |r|^2) - \hbar \frac{\partial \delta}{\partial E} |r|^2 + \frac{m|r|}{\hbar k_L^2} &\sin(2k_L x_L - \delta) . \end{aligned}$$

Using eqs. 15 and 14 in eq. 12 gives the dwell-time as

$$\begin{aligned}\tau_D &= \hbar \left(\frac{\partial \eta}{\partial E} |t|^2 + \frac{\partial \delta}{\partial E} |r|^2 \right) - \frac{\hbar |r|}{2E} \sin(2k_L x_L - \delta) \\ &+ \left(\frac{x_R}{v_R} - \frac{x_L}{v_L} \right) |t|^2 - \left(\frac{2x_L}{v_L} \right) |r|^2 . \end{aligned}\quad (16)$$

Since the potential lies between the limits, we can assume that x_L is a negative distance, while x_R is positive. The terms on the second line have an obvious interpretation as the “free passage time”, so the top line is the “dwell-time delay”. But be careful: Nussenzweig [5] has a lucid discussion of the case of an incident wave-packet, and finds an additional effect which arises from the uncertainty principle: you cannot localize a quantum particle in less than a de Broglie wave length. This effect cancels out between the free and interacting situations, so it affects dwell-time delay, but not dwell-time. We skip over this complication and rely on eq. 16.

Phase time was introduced by Wigner and Eisenbud [28], and similarly relates time delay to the rate of change of the scattering phase. Scattering in 1D is a two-channel

problem (incident waves from left, or right). The S -matrix is symmetric and unitary, and may be written

$$S = \begin{pmatrix} r & t \\ t & \bar{r} \end{pmatrix} = \begin{pmatrix} |r| e^{i\delta} & |t| e^{i\eta} \\ |t| e^{i\eta} & -|r| e^{i(2\eta - \delta)} \end{pmatrix} \quad (17)$$

where $\bar{r} = -r^* t / t^*$ is the reflection amplitude for waves incident from the right. $|r|^2 + |t|^2 = 1$.

Smith [29] defined the time-delay matrix for a many-channel system as

$$\begin{aligned}\tau &\equiv -i\hbar S^\dagger \frac{dS}{dE} = -i\hbar S^\dagger \times \\ &\begin{pmatrix} (|r'| + i|r|\delta') e^{i\delta} & (|t'| + i|t|\eta') e^{i\eta} \\ (|t'| + i|t|\eta') e^{i\eta} & -(|r'| + i|r|(2\eta' - \delta')) e^{i(2\eta - \delta)} \end{pmatrix} \end{aligned}\quad (18)$$

where primes mean derivative with respect to energy. After some work we have

$$\begin{aligned}\tau_{11} &= +\hbar \left(|t|^2 \frac{d\eta}{dE} + |r|^2 \frac{d\delta}{dE} \right) \\ \tau_{12} &= -i\hbar e^{i(\eta - \delta)} \frac{\partial}{\partial E} \left(\tan^{-1} \left(\frac{|r|}{|t|} \right) \right) + \hbar r^* t \left(\frac{d\eta}{dE} - \frac{d\delta}{dE} \right) \\ \tau_{22} &= +\hbar \left(\frac{d\eta}{dE} + |r|^2 \left(\frac{d\eta}{dE} - \frac{d\delta}{dE} \right) \right) . \end{aligned}\quad (19)$$

In the case of a reflection symmetric potential, $\bar{r} = r$, which requires $\eta = \delta + \pi/2$, so their derivatives are equal. As a result, all elements τ_{ij} are real, and the diagonal elements are equal. For a reflection-symmetric potential we have the compact result

$$\begin{aligned}\tau_{11} &= \hbar \frac{d\eta}{dE} = \tau_{22} \\ \tau_{12} &= \hbar \frac{\partial}{\partial E} (\sin^{-1} |r|) = \hbar (|r'|/|t|) . \end{aligned}\quad (20)$$

The diagonal elements are called the “phase-time delay”. They agree with the first term in the top line of eq. 16. But due to coupling, the time associated with a process depends on just which mixture of the two channels is involved. The difference is that eq. 16 was derived assuming that one has specified the channel with incident waves from the left, ignoring the other channel. The oscillatory term of eq. 16 has been subject of much debate also. Obviously it arises from interference between the incident and reflected waves. Winful [31] interpreted this term as the time taken to cross a distance of the order of the scattering length. On a more practical note, if $x_L \rightarrow -\infty$, and one averages k_L over a wave packet narrow in energy, this oscillatory term is exponentially small. Therefore in a practical sense, either dwell-time or phase-time are equivalent for SL scattering, at least for symmetric potentials.

Eq. 20 shows that there are two situations where the channel coupling vanishes: at a maximum or minimum of transmission. In a SL, the number of maxima is $N - 1$, so as N increases, these points of uncoupling become closer together.

B. A tale of two phases

In the transfer matrix method, the wave functions are defined with a different phase than in usual scattering theory: we reset the phase to zero on each side ($x = a, b$) of the potential array, rather than at an arbitrary origin. That makes M translation invariant, so M^N describes a periodic system without having to include a shift in the origin.

Adopting Nussenzweig's notation $[\psi_L; \psi_R]$ for the asymptotic wave function to left and right [5], we compare our $\psi(x)$ with the usual convention $\tilde{\psi}(x)$ as follows:

$$\begin{aligned}\psi(x) &\sim [e^{ik(x-a)} + re^{-ik(x-a)}; te^{ik(x-b)}] \\ \tilde{\psi}(x) &\sim [e^{ikx} + \tilde{r}e^{-ikx}; \quad \tilde{t}e^{ikx}] \\ e^{ika} \psi(x) &\sim [e^{ikx} + re^{-ik(x-2a)}; \quad te^{ik(x-w)}] ,\end{aligned}\quad (21)$$

where $w = b - a$ is the total width of the potential. It follows that the phase η of our transmission amplitude is related to the usual phase by $\tilde{\eta} = \eta - kw$. Then the standard phase-time delay is

$$\tau_{ph} = \hbar \frac{d\tilde{\eta}}{dE} = \hbar \left[\frac{d\eta}{dE} - w \frac{dk}{dE} \right] = \hbar \left[\frac{d\eta}{dE} - \frac{w}{v_{cl}} \right]. \quad (22)$$

We conclude that the phase η of our transfer matrix amplitudes gives phase time, not time-delay, because the free-passage time has to be subtracted from it.

IV. PHASE TIME FOR SUPERLATTICE TRANSMISSION

A. Relation of scattering phases to Kard parameters

The S -matrix and the transfer matrix contain the scattering information in different forms. To see how phase-time applies to a SL, we examine the relation between the two descriptions. Equating

$$\frac{1}{t_N} = \cos N\phi - i \sin N\phi \cosh \mu \equiv \frac{1}{|t_N|} e^{-i\eta_N} \quad (23)$$

we find the following relations between the two sets of parameters:

$$\begin{aligned}\cos \eta_N &= |t_N| \cos N\phi \quad ; \quad \sin \eta_N = |t_N| \sin N\phi \cosh \mu \\ \tan \eta_N &= \tan N\phi \cosh \mu \quad ; \quad \frac{\tan \eta_N}{\tan N\phi} = \cosh \mu = \frac{\tan \eta}{\tan \phi} ,\end{aligned}\quad (24)$$

where $\eta = \eta_1$ is the phase shift for a single cell. In the first allowed band, $0 < \phi < \pi$; $\tan \eta$ varies smoothly; $\cosh \mu$ diverges at the band edge, like $1/\sin \phi$. Zeroes and poles of $\tan \eta_N$ coincide with those of $\tan N\phi$. These points are the scattering resonances $N\phi_m = m\pi$ and the minima of transmission $N\phi_p = (p + 1/2)\pi$. Near the poles, η_N lags

behind $N\phi$, then it has to catch up at the zeroes $N\phi_m$. This is nicely illustrated in Fig. 5. The steeper slope of $/\eta_N$ near ϕ_m makes for a longer time delay at those energies.

B. Phase time near the maxima of transmission

In an infinite periodic array the electron would move at the Bloch velocity. Let the time to cross a cell of width d be τ_{Bl} . Write $\phi/d = \kappa$, the pseudo-momentum.

$$\begin{aligned}\frac{\partial E}{\partial \phi} &= \frac{1}{d} \frac{\partial E}{\partial \kappa} = \frac{\hbar^2 \kappa}{d m^*} = \frac{\hbar}{d} v_{Bl} = \frac{\hbar}{\tau_{Bl}} \\ \hbar \frac{\partial \phi}{\partial E} &= \tau_{Bl} .\end{aligned}\quad (25)$$

In the following, the prime will mean $\partial/\partial E$. We can show using eqs. 23 and 24 that

$$\begin{aligned}\frac{\partial \eta_N}{\partial E} &= \frac{[N\phi' \cosh \mu + \sin 2N\phi \sinh \mu \mu'/2]}{[1 + \sinh^2 \mu \sin^2 N\phi]} \\ \tau_{ph} &= N\tau_{Bl} \cosh \mu \frac{[1 + \sin 2N\phi \tanh \mu (\mu'/(2N\phi'))]}{[1 + \sinh^2 \mu \sin^2 N\phi]}\end{aligned}\quad (26)$$

Away from resonance, the denominator is a factor of $|t_N|^2$ which cuts off the phase time very sharply, causing it to mimic the shape of the transmission curve. Near a resonance, $\phi \sim m\pi/N + \varepsilon$, so that

$$(-)^m \sin N\phi \sim \sin N\varepsilon \sim N(E - E_m) \phi'_m .$$

This allows us to approximate $|t_N|^2$ as a Breit-Wigner resonance, with half-width $\Gamma_m = 2/(N \sinh \mu_m \phi'_m)$:

$$|t_N|_m^2 = \left[1 + \left(\frac{E - E_m}{\Gamma_m/2} \right)^2 \right]^{-1} . \quad (27)$$

Similarly, in the same vicinity,

$$\begin{aligned}\tau_{ph,m} &= \hbar \frac{\partial \eta_N}{\partial E} \Big|_m \sim N\tau_{Bl,m} \cosh \mu_m \times \\ &\quad \times \left[1 + 2b_m \left(\frac{E - E_m}{\Gamma_m/2} \right) \right] / \left[1 + \left(\frac{E - E_m}{\Gamma_m/2} \right)^2 \right] \\ \text{with } 2b_m &= \frac{1}{N\phi'_m \cosh \mu_m} \left[2\mu'_m + \coth \mu_m \frac{\phi''_m}{\phi'_m} \right]\end{aligned}\quad (28)$$

This is called a Fano resonance shape [30]. From eq. 26, we see that the locus of phase time at transmission maxima is

$$\tau_{ph,max} = N\tau_{Bl} \cosh \mu \quad (29)$$

Similarly, at transmission minima, $\sin^2 2N\phi = 1$; the denominator of τ_{ph} (eq. 26) becomes $\cosh^2 \mu$, giving a downside locus of

$$\tau_{ph,min} = N\tau_{Bl}/\cosh \mu \quad (30)$$

for phase time at transmission minima. The Bloch time for N cells is the geometric mean of the two loci.

C. Phase time near the minima

Transmission minima occur at $\phi_p = (p + 1/2)\pi/N$, for $p = 1, 2, \dots, n - 2$. Close by, the denominator may be written

$$\begin{aligned} t_N^{-2} &= \cosh^2 \mu (1 - \tanh^2 \mu \cos^2 N\phi) \\ &\sim \cosh^2 \mu_p [1 + \mu'_p \tanh \mu_p (\delta E_p)]^2 [1 - (N\phi'_p \tanh \mu_p \delta E_p)^2] \\ &= \cosh^2 \mu_p \left[1 + \frac{\mu'_p}{N\phi'_p} \left(\frac{\delta E_p}{\Gamma_p/2} \right) \right]^2 \left[1 - \left(\frac{\delta E_p}{\Gamma_p/2} \right)^2 \right] \end{aligned} \quad (31)$$

where $\delta E_p = E - E_p$. The width

$$\Gamma_p = 2/[N\phi'_p \tanh \mu_p] \quad (32)$$

is large compared to the widths Γ_m of the resonances.

$$\begin{aligned} \tau_{ph,p} &= \hbar \frac{\partial \eta_N}{\partial E} \Big|_p \\ &\sim \frac{N\tau_{Bl,p}}{\cosh \mu_p} \frac{\left[1 + C_p \left(\frac{E-E_p}{\Gamma_p/2} \right) \right]}{\left[1 + 2D_p \left(\frac{E-E_p}{\Gamma_p/2} \right) + (D_p^2 - 1) \left(\frac{E-E_p}{\Gamma_p/2} \right)^2 \right]} \end{aligned} \quad (33)$$

$$\text{where } C_p = \frac{\phi''_p}{\phi'_p} \frac{\Gamma_p}{2} \quad \text{and} \quad D_p = \frac{\mu'_p}{N\phi'_p}. \quad (34)$$

Both C_p and D_p are of order $1/N$. The prefactor is the locus of time-delay at minima:

$$\frac{N\tau_{Bl}}{\cosh \mu} = \frac{N\hbar}{\cosh \mu} \frac{d\phi}{dE} = -N\hbar \frac{d \cos \phi}{dE} / \text{Im} M_{11} \quad (35)$$

which involves only well-behaved single-cell quantities.

Veenstra et al. [18] compared their computed time delay to phase-time delay, and found good agreement. In particular they found that the locus of maxima and eqs. 27, 28 accounted for the overall picture. In an interesting paper, Pacher, Boxleitner and Gornik [14] pointed out that all theories of time-delay agree at the transmission maxima, so they concentrated their attention on those points and found the locus of maxima. The main difference between their work and the present one is that they spoke of the mean velocity for an electron traversing the SL, rather than the time. Further they expressed their results in terms of the real and imaginary parts of the transfer matrix elements M_{ij} , e.g. eq. 35, rather than the Kard parameters. It is our opinion that the Kard parameters, given their simple behaviour, make the expressions more easily understandable.

D. Play Model

To illustrate the above results, we take a simple model in which we specify $\cos \phi$ to be linear in energy in a band

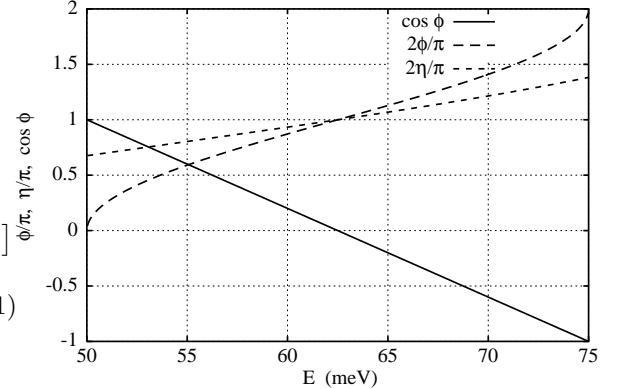


FIG. 2: Play model: $\cos \phi$, and angles ϕ , and η , in units of $\pi/2$. Remarkably, ϕ is reasonably linear over much of the band.

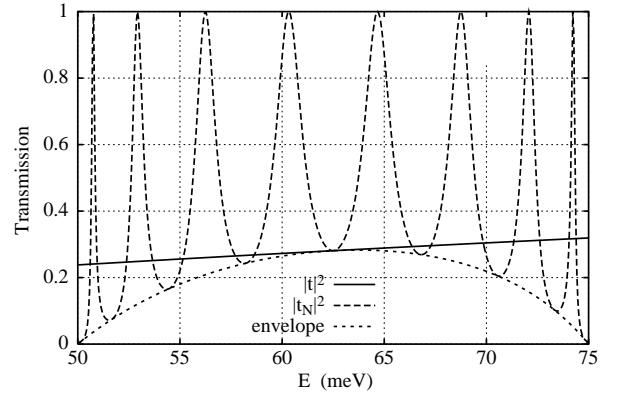


FIG. 3: Play model: Transmission probability for one cell, nine cells, and envelope of transmission minima.

between $E = 50$ and 75 meV. In addition, $|t|^2$ is specified. All other parameters follow from these two. With such a model it is easy to see how changing some parameter will affect the results.

$$\begin{aligned} \cos \phi &= 0.08(62.5 - E), \quad 50 \leq E \leq 75 \text{ (meV)}, \\ |t|^{-2} &= 1 + 160/E. \end{aligned} \quad (36)$$

The phase η is determined by $\cos \eta = |t| \sin \phi$, and the impedance parameter follows from

$$\cosh \mu = \sin \eta / (|t| \sin \phi). \quad (37)$$

It diverges at each band edge, since $|t| \sin \eta$ is a smooth function of energy, while ϕ runs from $p\pi$ to $(p+1)\pi$.

In Fig. 2 we show the play model $\cos \phi = \lambda(E_B - E)$, and the corresponding phases ϕ and η . η is quite linear, crossing ϕ at the Bragg point. In this model, $\phi' = \lambda/\sin \phi$, with $\lambda = 0.08 \text{ meV}^{-1}$.

In Fig. 3 we show the play model transmission for 9 cells. The envelope of transmission minima is simply $1/\cosh^2 \mu$, which gives direct physical significance to the impedance parameter.

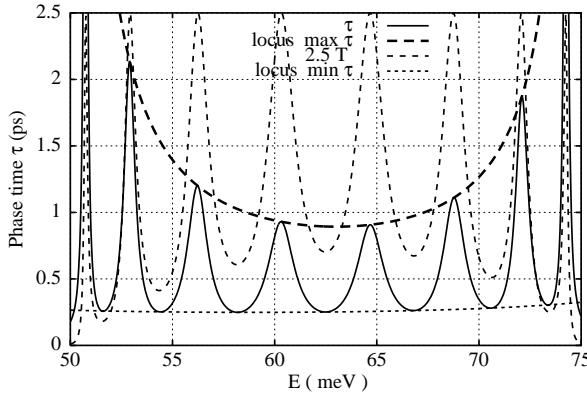


FIG. 4: Play model: phase time, and loci of phase time at maxima and minima of transmission. Transmission probability is shown for orientation.

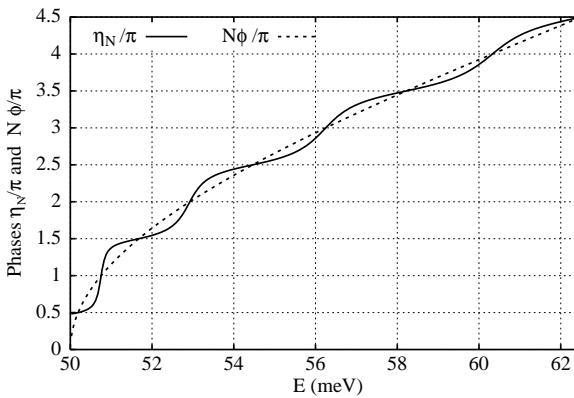


FIG. 5: Play model: η_N and $N\phi$, in units of π . Note that slope of η_N is largest at integer multiples of π . For clarity, only half the allowed band is shown.

In Fig. 4 we show the play model phase-time, along with the envelopes of maxima and minima. In the background for orientation are the transmission peaks, which line up well with the maxima of phase-time.

In Fig. 5 we show the dance of the Bloch phase for N cells, and the corresponding transmission phase η_N , in the lower half of the allowed band. The curve in the upper half of the band is a double reflection of this, ending at 9π . Except at the band edges, the lines cross at every half-integer multiple of π . Since η_N is catching up at integer multiples of π , the steeper slope leads to a longer phase-time at the transmission resonances.

In Fig. 6 we show (solid line) the transmission and (dashed line) the Breit-Wigner resonance fitted at the peaks. This is truncated at two standard deviations; the horizontal lines simply connect the B-W curves between successive peaks. The agreement is excellent.

Fig. 7 is similar to Fig. 6, but for the phase-time. The Fano-shape formula fitted to the maxima also does an excellent job of reproducing the exact calculation.

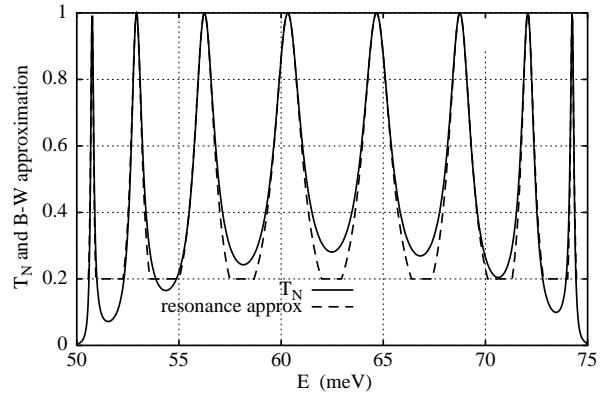


FIG. 6: Transmission probability compared to resonance approximation (eq. 27).

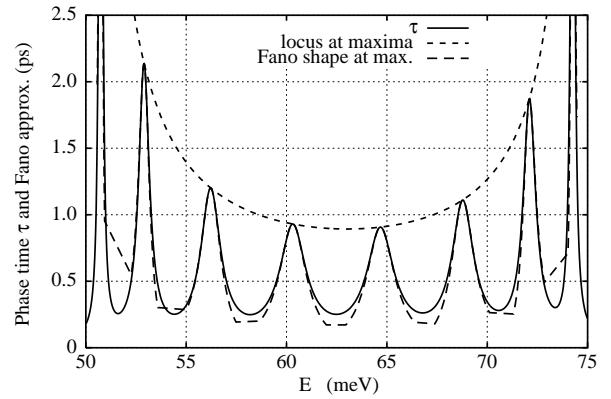


FIG. 7: Play model: phase time, and its approximation by resonance formula eq. 28. Locus of phase time at maximum transmission is shown to demonstrate that phase-time maxima are very little shifted from the locus.

E. Realistic potential model

Some results using a realistic semiconductor potential of Pacher and Gornik [23, 24] are shown in Figs. 8, 9. Here we took five potential cells, giving four resonances in the band, a number chosen for comparison with the calculations of Veenstra et al. [18]. Fig. 8 corresponds to Fig. 4 for the phase-time. For this potential the envelope of minima is lower meaning a larger μ . Also shown is the Bloch time, the geometric mean of the two envelopes.

Fig. 9 can be compared with Fig. 7. The Fano formula fitted at the maxima of transmission agrees with the exact result over two standard deviations. In Fig. 9 we have included eq. 33, fitted at the minima of transmission. The fit to the minima is not so good, largely because the half-width at minima is so much larger than at maxima. Still, we can say that between them the approximations reproduce the phase time over about 90% of the band. If there were more layers, the peaks and valleys would be narrower and the agreement would improve, as in Fig. 7. For large N the phase time can be

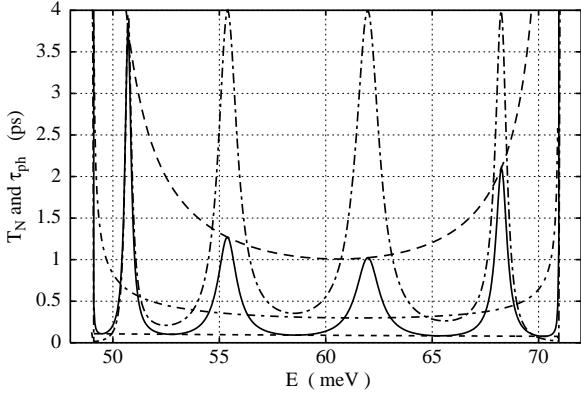


FIG. 8: Pacher-Gornik 5-cell array: phase time (solid line), along with the loci at maxima and minima (long dashes), and Bloch time (dash - dot), which is their geometric mean. The chain-line is four times the transmission probability.

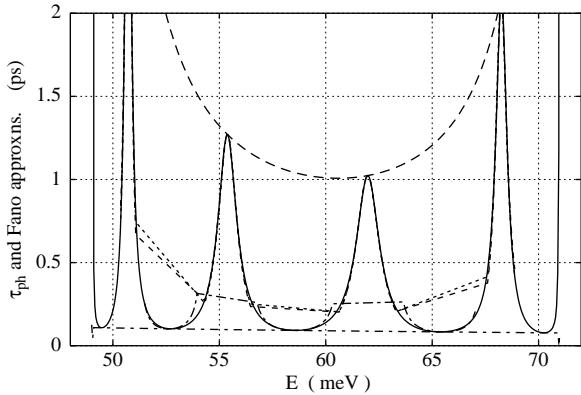


FIG. 9: Pacher-Gornik 5-cell array: along with the Fano-type approximation at each resonant peak and valley. The fit at the peaks is excellent over two half-widths, but not so good at the phase-time minima. The long dash and dash-dot lines are the loci of phase-time at maximum and minimum transmission.

calculated reliably using only properties at the extrema, where the various theories of time-delay agree with each other.

Finally, in Fig. 10 we show the results calculated by Veenstra et al. for the Pacher five cell array, plus a two layer ARC. The dotted line shows the transmission is close to 100% over most of the band width. There are two lines shown for the Bloch time. One takes into account only the periodic 5-cell system, and the other includes a correction for the ARC layers on each end. The two estimates are close together. The time delay extracted from the TDSE calculations is indeed very close to the Bloch time, which is what we expect from the argument that the ARC converts the incident plane wave into a Bloch state.

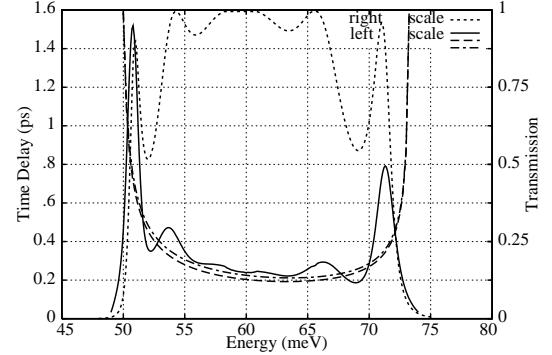


FIG. 10: Time-dependent numerical solution of wave equation for a Pacher-Gornik 5-cell array plus ARC: phase-time compared to Bloch time. Transmission shown in background.

V. CONCLUSION

We have outlined the transfer matrix method for transmission of electrons in a one-dimensional AlGaAs/GaAs superlattice, using the Kard parameterization. We then compared the two most commonly used theories for the average time spent in crossing a potential region, namely phase time and dwell time. These were applied to SL transmission, first for a play model and then for the potential that corresponds to experiments of Pacher and Gornik. We noted that the phase-time is well defined both at maxima and minima of transmission. In the neighbourhood of these points, the transmission can be described by a Breit-Wigner resonance formula, with the parameters extracted at the extrema. The same holds for the phase-time, except it is a Fano shape resonance. The fits at maxima are good over two half-widths, and at the minima over one. Taken together, this covers almost all the band width, for a system with more than a very few periods.

As the number of periods N of the SL is increased, the widths of the peaks and valleys decrease, and the approximate forms become more and more accurate. This explains the success of phase-time for describing the time spent in traversing a SL.

The locus of phase-time at maxima was derived in Veenstra et al. [18], and by Pacher et al. [14]. The locus of phase-time at mimima is new. Their geometric average is the Bloch time for traversing a SL. An ARC works by converting the incident plane wave into a Bloch wave of the periodic system. Veenstra et al. [18] studied the time delay for the SL plus ARC by direct numerical solution of the TDSE using gaussian wave packets. In Fig. 10, taken from that reference, it can be seen that indeed the time-delay with ARC agrees quite well with the Bloch time. The ARC not only increases the average transmission, but it smooths out the dwell time, removing the peaks and valleys associated with the exponential decay of the resonances.

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